LPV System Common State Basis Estimation from Independent Local LTI Models ⋆

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Abstract: For the identification of a linear parameter varying (LPV) system steered by a scheduling variable evolving within a finite set, the local approach consists in separately estimating local linear time invariant (LTI) models corresponding to fixed values of the scheduling variable. It is shown in this paper that, without any global structural assumption of the considered LPV system, the local state-space LTI models do not contain the necessary information about the similarity transformations making them coherent. Nevertheless, it is possible to estimate these similarity transformations from input-output data under appropriate input excitation conditions. These estimations result in a common state basis of the transformed local LTI models, so that they form a coherent global LPV model, suitable for numerical simulations in the case of fast scheduling variable evolutions.

Keywords: System identification, LPV model, coherent local linear models.

1. INTRODUCTION

Linear parameter varying (LPV) models provide an effective approach to handling nonlinear control systems (Toth, 2010; Mohammadpour and Scherer, 2012; Lopes dos Santos et al., 2012; Sename et al., 2013). Some successful methods for LPV system identification have been reported recently (Van Wingerden and Verhaegen, 2009; Mercere et al., 2011; Lopes dos Santos et al., 2011; Toth et al., 2012; Zhao et al., 2012; Piga et al., 2015), with various assumptions about LPV model structures. As a matter of fact, many variant LPV model structures have been proposed, and the methods developed for LPV system identification strongly depend on the particular model structures.

The present paper is focused on the identification of LPV systems in the case where the scheduling variable $p(t)$ takes values from a finite set, say $P = \{p_1, \dots, p_m\}$. Typically each $p_i \in P$ corresponds to a working point of the considered system. It is assumed that input-output data are collected at different working points, and that the resulting LPV model will be used at the same working points. In other words, the problem of local model interpolation between working points, as studied in (De Caigny et al., 2011, 2014), is not considered in the present paper.

With the scheduling variable $p(t)$ restricted to a finite set P, the considered particular class of LPV systems will be referred to as *linear finite parameter varying* (LFPV) systems. When $p(t)$ is fixed to a particular value, the related LFPV system behaves like a linear time invariant (LTI) system. An LFPV system is thus characterized by a collection of *local LTI models*.

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Given an input-output data sample and the corresponding scheduling variable sequence, in principle the LFPV system identification problem can be solved by the prediction error method (PEM) after having chosen some parametrization of the local LTI models. However, such a solution implies solving a large optimization problem, in terms of the number of unknowns and the amount of data to be processed as a whole. Moreover, it is not easy to make a good initial guess of the model parameters before applying the PEM. Methods following such a global approach often assume some global parametric structure of the LPV system (Toth, 2010).

Alternatively, it is possible to separately estimate each local LTI model from the data collected at the related working point corresponding to a particular scheduling value $p_i \in P$. Such a local approach is attractive in practice, because the whole LFPV model can be built and completed progressively, by limiting the requirement on computing resources. Unless physical models are used, each local state-space LTI model is estimated up to an arbitrary similarity transformation. If each estimated local model is only used at the related working point (in this case the concept of LFPV model is of little interest), the indetermination of the arbitrary similarity transformation is not a problem at all. However, if the estimated collection of local LTI models is used as a global LFPV model spanning different working points, the estimated local LTI models must be *coherent*, in the sense that they are related to the true LFPV system by the *same* similarity transformation.

Global structural assumptions about the LFPV system can help to make the estimated local LTI models coherent (see examples in Section 3.2). Such assumptions should be based on physical insights about the considered system, otherwise they may excessively restrict the flexibility of the resulting model structure. In practice, local LTI models are often reduced order approximation a complex system, then such models have a strong black-box nature. In this case it is difficult to make the estimated local LTI models coherent.

The main purpose of this paper is to point out the fact that, *without global structural assumptions*, the local LTI models themselves do *not* contain the necessary information about the similarity transformations making them coherent. Nevertheless, it is possible to estimate these similarity transformations from input-output data under some excitation conditions, through the estimation of the states around the scheduling variable jump instants, with a method initially introduced in the framework of piecewise linear hybrid systems (Verdult and Verhaegen, 2004).

2. PROBLEM STATEMENT

Let $u(t) \in \mathbb{R}^{n_u}$ and $y(t) \in \mathbb{R}^{n_y}$ be respectively the input and the output of a dynamic system at discrete time instants $t \in \mathbb{N}^* = \{0, 1, 2, \dots\}$. Assume that there exists a scheduling variable $p(t)$ defined by $p : \mathbb{N}^* \to P$, where P is a finite set

$$P = \{p_1, p_2, \dots, p_m\}, \tag{1}$$

such that the considered dynamic system is described by the finite dimensional state-space model

$$x(t+1) = A(p(t))x(t) + B(p(t))u(t) + w(t) \tag{2a}$$

$$y(t) = C(p(t))x(t) + D(p(t))u(t) + v(t) \tag{2b}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $A(p(t)), B(p(t)), C(p(t)), D(p(t))$ are matrices of appropriate sizes depending on $p(t) \in P$, and $w(t) \in \mathbb{R}^{n_x}, v(t) \in \mathbb{R}^{n_y}$ are white Gaussian noises with covariance matrices $Q(p(t)), R(p(t))$. Given a scheduling sequence $p(t)$, an input sequence $u(t)$, a realization of the random noises $w(t), v(t)$, and an initial state $x(0) = x_0 \in \mathbb{R}^{n_x}$, the state $x(t)$ and output $y(t)$ of the system are then fully determined.

Consider a set of consecutive time instants

$$\mathbb{T}_l^k = \{l, l+1, l+2, \dots, k\} \tag{3}$$

within which the value of $p(t)$ is fixed, say $p(t) = p_i$ for all $t \in \mathbb{T}_l^k$. Within these time instants, the system formulated by (2) is characterized by constant matrices $A(p(t)), B(p(t)), C(p(t)), D(p(t)), Q(p(t)), R(p(t))$, say

$$A_i = A(p_i),\ B_i = B(p_i), C_i = C(p_i),\ \text{etc.}, \tag{4}$$

corresponding to a linear time invariant (LTI) system

$$x(t+1) = A_i x(t) + B_i u(t) + w(t) \tag{5a}$$

$$y(t) = C_i x(t) + D_i u(t) + v(t) \tag{5b}$$

for $t \in \mathbb{T}_l^k$. Such a model will be referred to as a *local LTI model*, as it is valid only at the working point specified by $p(t) = p_i$. The notation

$$\text{LTI}_i \triangleq (A_i, B_i, C_i, D_i, Q_i, R_i) \tag{6}$$

will be used to denote the local LTI system model indexed by i.

In addition to the local LTI behavior of the system when the value of $p(t)$ remains constant, the model formulated in (2) specifies also the transition at every jump of $p(t)$. Assume that $p(k) = p_i \neq p(k+1) = p_j$. During this transition the system is no longer LTI, because of the changes of the matrices $A(p(t)), B(p(t))$ etc.. Nevertheless, according to (2), the value of $x(k+1)$ is determined by

$$x(k+1) = A_i x(k) + B_i u(k) + w(k). \tag{7}$$

The whole trajectory of $x(t)$ is thus well defined for all $t \in \mathbb{N}^*$.

As the scheduling variable $p(t)$ evolves within the finite set P, the matrix function $A(p)$ takes also values within a finite set of matrices, say $\{A_1, \dots, A_m\}$, and so do similarly the other system matrices. The characteristics of a system as formulated in (2) are thus fully specified by the finite sets of matrices $\{A_1, \dots, A_m\}$, $\{B_1, \dots, B_m\}$, etc., without requiring any structural assumption about the matrix functions $A(p), B(p)$, etc..

Based on the above comments, the global system described by (2) with $p(t) \in P$ will be referred to as a *linear finite parameter varying* (LFPV) system (the word "finite" refers to the fact that P is a finite set).

The LFPV system identification problem considered in this paper is to estimate the finite sets of matrices $\{A_1, \dots, A_m\}$, $\{B_1, \dots, B_m\}$, etc., *solely* from the scheduling sequence $p(t)$, the input-output data $u(t), y(t)$ for $t = 0, 1, \dots, N$, and the known model order n_x.

When the local models LTI_i are separately estimated without using any global structural information about the LFPV system, each local model LTI_i can only be estimated *up to a similarity transformation*. However, to ensure consistent *global* simulation with an estimated LFPV model, the estimated local LTI models must be related to the true LFPV system by *the same* similarity transformation. This fact motivates the following definition.

Definition 1. The set of local LTI models

$$\{(\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i, \tilde{Q}_i, \tilde{R}_i) :\ i = 1, 2, \dots, m\} \tag{8}$$

constitutes a *coherent* representation of the LFPV system (2) composed of a set of local LTI systems as formulated in (5) and characterized by $(A_i, B_i, C_i, D_i, Q_i, R_i)$, if there exits an invertible transformation matrix $T \in \mathbb{R}^{n_x \times n_x}$ such that, for all $i = 1, \dots, m$,

$$\tilde{A}_i = T A_i T^{-1},\ \tilde{B}_i = T B_i, \tag{9a}$$

$$\tilde{C}_i = C_i T^{-1},\ \tilde{D}_i = D_i, \tag{9b}$$

$$\tilde{Q}_i = T Q_i T^T,\ \tilde{R}_i = R_i. \tag{9c}$$

□

In practice, when a set of local LTI models are estimated from a finite data sample subject to random uncertainties, the definition of coherent local models should be understood in an approximative sense.

If some global structural assumptions of the matrix functions $A(p), B(p)$, etc. were assumed, then it would be relatively easy to make estimated local LTI models coherent, as given $A(p_i) = A_i, B(p_i) = B_i$, etc. for any $i = 1, 2, \dots, m$, the other $A(p_j) = A_j, B(p_j) = B_j$, etc. would be partly or fully determined. Unless based on particular physical insights, such global structural assumptions may excessively restrict the flexibility of the LFPV model structure. In this paper, the LFPV system identification problem is considered with a fully flexible LFPV model structure characterized by a set of *independent local LTI models*, as defined below.

Definition 2. A set of local LTI models composing an LFPV system, as formulated in (5), are *independent* if their parametrizations are such that the matrices A_i, B_i, C_i, Q_i characterizing LTI_i do not imply any information about the matrices A_j, B_j, C_j, Q_j characterizing LTI_j, for all $i \neq j$, both belonging to $\{1, 2, \dots, m\}$. □

Remark 1. In principle, it is possible to apply the prediction error method (PEM) (Ljung, 1999) to simultaneously estimate all the finite sets of matrices $\{A_1, \dots, A_m\}$, $\{B_1, \dots, B_m\}$, etc. by processing all the available data as a whole, but such a global approach amounts to processing a large set of data as a whole, and requires reasonably coherent initial guesses of the local LTI models. Alternatively, this paper follows a local approach, by processing the available data in pieces segmented according to the value of $p(t)$, without requiring initial guesses of local LTI models.

Remark 2. In this paper, the absence of any *global structural assumption* implies the LFPV model structure composed of *independent* local LTI models in the sense of Definition 2. This is a major difference from most LPV system identification methods which are based on some particular global structural assumptions, typically with a affine structure and a reduced set of parameters (Toth, 2010; Lopes dos Santos et al., 2012).

3. ATTEMPTS TO MAKING LOCAL LTI ESTIMATES COHERENT

Compared to the global approach, the advantages of the local approach have been mentioned in Remark 1 of the previous section. However, the local approach has also a serious problem: as the local models estimated from input-output data are expressed in arbitrary state bases, the resulting local models LTI_i are in general not coherent in the sense of Definition 1. A set of incoherent local LTI models cannot be used together as a whole LFPV model.

3.1 A general fact about independent local LTI models

Proposition 1. Assume that the local systems LTI_i as formulated in (5) composing a true LFPV system are *independent* in the sense of Definition 2. Given a set of LTI models ($\hat{A}_i, \hat{B}_i$, etc.) such that

$$\hat{A}_i = \hat{T}_i A_i \hat{T}_i^{-1},\ \hat{B}_i = \hat{T}_i B_i, \text{etc.} \tag{10}$$

for $i = 1, \dots, m$, where $\hat{T}_i \in \mathbb{R}^{n_x \times n_x}$ are *arbitrary unknown* invertible matrices, then it is *impossible* to determine similarity transformation matrices $\tilde{T}_i \in \mathbb{R}^{n_x \times n_x}$ for $i = 1, \dots, m$, *solely* based on the given LTI models ($\hat{A}_i, \hat{B}_i$, etc.) themselves, so that the transformed LTI models characterized by

$$\tilde{A}_i = \tilde{T}_i \hat{A}_i \tilde{T}_i^{-1},\ \tilde{B}_i = \tilde{T}_i \hat{B}_i, \text{etc.}, \tag{11}$$

are coherent in the sense of Definition 1; or in other words, the involved unknowns are *underdetermined* by the available equations. □

Proof. The proof consists in counting the unknowns and the available equations.

All the relevant equations are, for $i = 1, \dots, m$,

$$\tilde{A}_i = \tilde{T}_i \hat{A}_i \tilde{T}_i^{-1},\ \tilde{B}_i = \tilde{T}_i \hat{B}_i,\ \tilde{C}_i = \hat{C}_i \tilde{T}_i^{-1},\ \tilde{Q}_i = \tilde{T}_i \hat{Q}_i \tilde{T}_i^T$$
$$\tilde{A}_i = T A_i T^{-1},\ \tilde{B}_i = T B_i,\ \tilde{C}_i = C_i T^{-1},\ \tilde{Q}_i = T Q_i T^T.$$

where $\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{Q}_i$ are known, whereas all the other involved quantities are unknowns.

To take into account the fact that there is no need to uniquely determine the transformation matrix T common to all $i = 1, \dots, m$, eliminate $\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{Q}_i$ from these equations and replace the unknowns $\tilde{T}_i, T$ by $\tilde{\tilde{T}}_i \triangleq T^{-1}\tilde{T}_i$ for every $i = 1, \dots, m$. Then the remaining equations are

$$A_i = \tilde{\tilde{T}}_i \hat{A}_i \tilde{\tilde{T}}_i^{-1},\ B_i = \tilde{\tilde{T}}_i \hat{B}_i,\ C_i = \hat{C}_i \tilde{\tilde{T}}_i^{-1},\ Q_i = \tilde{\tilde{T}}_i \hat{Q}_i \tilde{\tilde{T}}_i^T$$

with the unknowns A_i, B_i, C_i, Q_i and $\tilde{\tilde{T}}_i$ for $i = 1, \dots, m$.

If $\tilde{\tilde{T}}_i$ were known, then there would be exactly the same number of equations as the unknowns A_i, B_i, C_i, Q_i, either counted in the matrix sense or in the scalar sense. Because of the extra unknowns $\tilde{\tilde{T}}_i$, the entire unknowns are then clearly *underdetermined* by the whole set equations. Proposition 1 is thus proved. □

3.2 Examples using global structural assumptions

The result of Proposition 1 may seem in contradiction with some known publications proposing methods for making local LTI models coherent. In fact, each of these existing methods assumes, explicitly or implicitly, some particular structure of the matrix-valued functions $A(p), B(p)$ etc., therefore they do not cover the case studied in this paper. To better clarify the situations, some examples of the published methods are recalled in this subsection, by pointing out their particular global structural assumptions.

Coherent LTI models based on canonical forms

In order to make local LTI models coherent, a natural method is to find the similarity transformations leading to some canonical state-space form of the LTI systems, typically the controllable or the observable form. The idea behind this method is that the local models should be coherent when they are all transformed into the same canonical form.

For the sake of presentation simplicity, consider a single-input-single-output (SISO) LFPV system. In the controllable form, the m local models involve, for $i = 1, 2, \dots, m$,

$$\tilde{A}_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_i^{(1)} & a_i^{(2)} & a_i^{(3)} & \cdots & a_i^{(n)} \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{12}$$

The assumption that the local LTI models in their canonical form are coherent implies that there exists a single invertible matrix $T \in \mathbb{R}^{n_x \times n_x}$ such that, for all $i = 1, 2, \dots, m$, $\tilde{A}_i = T A_i T^{-1}$. Let $S(M)$ denote the sub matrix of M excluding its last row, then

$$S\left(T A_i T^{-1}\right) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad \forall i = 1, 2, \dots, m. \tag{13}$$

The assumption that $S\left(TA_iT^{-1}\right)$ are equal to the same particular matrix for all $i = 1, 2, \ldots, m$ is indeed a strong global structural assumption about $A(p)$.

Coherent LTI models based on the observability matrix

In (De Caigny et al., 2014) another method is proposed to make local LTI models coherent. The class of systems considered in (De Caigny et al., 2014) is more general than that of the present paper. When applied to the LFPV systems as considered in the present paper, this method consists in finding different similarity transformations so that the m transformed LTI models all have the same observability matrix.

This method is based on the assumption that the local LTI systems composing the considered LFPV system all have the same observability matrix, or more explicitly,

$$C_iA_i^s = C_jA_j^s \tag{14}$$

for all $i, j \in \{1, 2, \ldots, m\}$ and $s \in \{0, 1, \ldots, n_x - 1\}$. This is clearly also a global structural assumption about $A(p), C(p)$.

Notice that this observability matrix-based method is incompatible with the previously presented canonical form-based method, as in general the local LTI models in their canonical form do not have the same observability matrix. This incompatibility between the two "natural" methods confirms the fact that there is no generally natural global structural assumption for making estimated local LTI models coherent.

4. DATA-BASED TRANSFORMATIONS FOR COHERENT LTI MODELS

It was shown in the previous section that, in the local approach to LFPV system identification, it is impossible to determine similarity transformations to make independent local LTI models coherent *solely from the estimated local LTI models themselves.* It is thus necessary to make use of other information, not contained in the local LTI models. By excluding global structural assumptions, the only information left seems the available input-output data and the scheduling variable sequence. These data have already been used for the estimation of local LTI models, but they can also provide more information, for making the local models coherent.

4.1 The case of $m = 2$

For the ease of presentation, let us first consider the case of a LFPV system with only 2 local LTI systems ($m = 2$). The more general case will be considered later.

Assume that $p(t)$ changes from p_1 to p_2 at $t = k$, or more accurately, $p(t) = p_1$ for $t \in \mathbb{T}_l^k = \{l, l+1, l+2, \ldots, k\}$ and $p(t) = p_2$ for $t \in \mathbb{T}_{k+1}^q = \{k+1, k+2, \ldots, q\}$. Assume further that the two corresponding input-output data segments are informative enough so that two local LTI models of order n_x can be estimated from them, with any of the classical LTI system identification methods (Ljung, 1999). These estimated local models will be denoted by

$$\mathrm{LTI}_1 = (\hat{A}_1, \hat{B}_1, \text{etc.}) \text{ and } \mathrm{LTI}_2 = (\hat{A}_2, \hat{B}_2, \text{etc.}). \tag{15}$$

With the two estimated LTI models and the input-output data, the state sequence $x(t)$ for $t \in \mathbb{T}_l^k$ and $t \in \mathbb{T}_{k+1}^q$ can be estimated with different approaches. For some subspace identification methods, the state sequence estimate $\hat{x}(t)$ is a co-product (Van Overschee and De Moor, 1996). With the PEM (Ljung, 1999), the initial state estimate (for each of $\mathbb{T}_l^k, \mathbb{T}_{k+1}^q$) is also provided with the estimated LTI model, then the whole state sequence, separately for $t \in \mathbb{T}_l^k$ and $t \in \mathbb{T}_{k+1}^q$, can be estimated by the Kalman filter.

Here the final state estimate with LTI_1 for $t \in \mathbb{T}_k^k$, namely $\hat{x}_1(k)$, and the initial state estimate with LTI_2 for $t \in \mathbb{T}_{k+1}^q$, namely $\hat{x}_2(k+1)$, are of particular interests.

If LTI_1 and LTI_2 were coherent, then according to (7), the equality

$$\hat{x}_2(k+1) = \hat{A}_1\hat{x}_1(k) + \hat{B}_1u(k) \tag{16}$$

would hold, up to random estimation errors of $\hat{x}_1(k)$ and $\hat{x}_2(k+1)$ and the noise $w(k)$. In general, of course, LTI_1 and LTI_2 are not coherent and equality (16) does not hold.

Assume that after applying some similarity transformation $T_{1,2}$ to LTI_1, the transformed LTI model will be coherent with LTI_2. *The remaining part of this subsection* is for the purpose of introducing a method for the estimation of $T_{1,2}$.

As the application of the transformation matrix $T_{1,2}$ makes LTI_1 coherent with LTI_2, the incorrect equality (16) should be replaced by

$$\hat{x}_2(k+1) = T_{1,2}\hat{A}_1T_{1,2}^{-1}T_{1,2}\hat{x}_1(k) + T_{1,2}\hat{B}_1u(k) \tag{17}$$

$$= T_{1,2}\hat{A}_1\hat{x}_1(k) + T_{1,2}\hat{B}_1u(k). \tag{18}$$

Define

$$\hat{x}_1(k+1) \triangleq \hat{A}_1\hat{x}_1(k) + \hat{B}_1u(k) \tag{19}$$

as an estimate of $x(k+1)$ from LTI_1, then (18) is rewritten as

$$\hat{x}_2(k+1) = T_{1,2}\hat{x}_1(k+1). \tag{20}$$

As $\hat{x}_2(k+1)$ and $\hat{x}_1(k+1)$ can be both estimated from available data with the estimated local LTI models, they provide information about $T_{1,2}$ through (20).

The matrix $T_{1,2}$ has $n_x \times n_x$ unknown entries, but (20) contains only n_x scalar equations. It is not yet sufficient to determine $T_{1,2}$ in order to make the two local LTI models coherent.

Now assume that in the available data set there are more than one jumps of $p(t)$ within $P = \{p_1, p_2\}$. If each of these jumps leads to an equation on $T_{1,2}$ similar to (20), then it is possible to determine $T_{1,2}$ from these equations.

Assume that $k^{(0)}, k^{(1)}, k^{(2)}, \ldots, k^{(s+1)}$ are jump instants interlacing $p(t) = p_1$ and $p(t) = p_2$, such that for every

$$t \in \mathbb{T}_{k^{(j)}+1}^{k^{(j+1)}} = \{k^{(j)}+1, k^{(j)}+2, \ldots, k^{(j+1)}\}, \tag{21}$$

$p(t) = p_1$ if j is an odd number, and $p(t) = p_2$ if j is an even number.

If it was chosen to estimate separately one local LTI model from the data within each $\mathbb{T}_{k^{(j)}+1}^{k^{(j+1)}}$, those corresponding to odd numbers j would in principle all describe the same local LTI model, but in different state bases. To avoid this trouble, a simple idea is to treat all these data for $t \in \mathbb{T}_{k^{(j)}+1}^{k^{(j+1)}}$ with odd numbers j as a whole *multi-experiment* data set (Ljung, 2014), composed of different experiments corresponding to different $\mathbb{T}_{k^{(j)}+1}^{k^{(j+1)}}$. A single LTI model, namely LTI_1, is then estimated from this multi-experiment data set. Similarly, a single model LTI_2 is estimated from the multi-experiment data set corresponding to even numbers j. A similar method was proposed in (Verdult and Verhaegen, 2004) in the framework of subspace system identification.

With the two estimated LTI models $\mathrm{LTI}_1, \mathrm{LTI}_2$ and the available data subsets, after every jump instant $k^{(j)}$ for $j = 1, 2 \ldots, s$, two state estimates $\hat{x}_1(k^{(j)}+1)$ and $\hat{x}_2(k^{(j)}+1)$ are computed, respectively with LTI_1 and LTI_2.

When j is an even number, $\hat{x}_1(k^{(j)}+1)$ is computed from the final state estimate within $\mathbb{T}_{k^{(j-1)}+1}^{k^{(j)}}$ in a way similar to (19), whereas $\hat{x}_2(k^{(j)}+1)$ is simply the initial state estimate within $\mathbb{T}_{k^{(j)}+1}^{k^{(j+1)}}$.

When j is an odd number, $\hat{x}_1(k^{(j)}+1)$ is simply the initial state estimate within $\mathbb{T}_{k^{(j)}+1}^{k^{(j+1)}}$, whereas the computation of $\hat{x}_2(k^{(j)}+1)$ is made in a way similar to (19).

With these results, the vector equation (20) is generalized to the matrix equation

$$\begin{aligned}&[\hat{x}_2(k^{(1)}+1),\ \cdots,\ \hat{x}_2(k^{(s)}+1)]\\&= T_{1,2}[\hat{x}_1(k^{(1)}+1),\ \cdots,\ \hat{x}_1(k^{(s)}+1)].\end{aligned} \qquad (22)$$

Assume that the matrix $\hat{X}_1 \triangleq [\hat{x}_1(k^{(1)}+1),\ \cdots,\ \hat{x}_1(k^{(s)}+1)]$ has full row rank, then $T_{1,2}$ can be estimated, by simply inverting $\hat{X}_1$ if it is a square matrix, otherwise by solving (22) for $T_{1,2}$ in the least squares sense.

Remark that, for estimating two LTI models from the two multi-experiment data sets (Ljung, 2014) formed from the available data set, it is sufficient to assume that each of these two multi-experiment data sets are informative enough, instead of assuming that each of the data segment corresponding to $\mathbb{T}_{k^{(j)}+1}^{k^{(j+1)}}$ is informative enough.

4.2 The case of $m \geq 2$

Now consider the case of $m \geq 2$. The m local models LTI_i are first estimated from the available data, so is the whole state sequence.

For each pair of indexes $i \neq j$, both belonging to $\{1, 2, \ldots, m\}$, it is possible to estimate a transformation matrix $T_{i,j}$ to make the estimated LTI_i and LTI_j coherent, by applying the method described in the previous subsection to appropriately selected data segments.

If a transformation matrix was estimated for each pair of the m estimated local LTI models, there would be $m(m+1)/2$ such estimated transformation matrices, but they are not all necessary.

For example, assume that the available data set allows the estimation of the $m-1$ transformation matrices $T_{1,2}, T_{2,3}, \ldots, T_{m-1,m}$, then by applying the matrix product $T_{m-1,m} \cdot T_{m-2,m-1} \cdots T_{2,3} \cdot T_{1,2}$ as a single transformation matrix to LTI_1, the transformed model is coherent with LTI_m. Similarly, the other local models $\mathrm{LTI}_2, \ldots, \mathrm{LTI}_{m-1}$ are then also made coherent with LTI_m. Therefore, $m-1$ estimated transformation matrices are sufficient to make the m local LTI models coherent.

Generally, this method is based on the following assumptions.

A1. For each $p_i \in P$, the data subset
$$Z_i = \{(u(t), y(t), p(t)) : \ t = 0, 1, \ldots, N, \ p(t) = p_i\}$$
forms a multi-experiment data set sufficiently informative for the estimation of a state-space LTI model of order n_x, namely LTI_i.

A2. There are sufficient jumps in the scheduling sequence $p(t)$ such that pairs of adjacent data segments allow the estimation of $m-1$ transformation matrices $T_{i,j}$, each linking two local models LTI_i and LTI_j, so that the entire m local LTI models are linked together directly or indirectly by the of $m-1$ transformation matrices.

The complete algorithm for estimating the $m-1$ transformation matrices is well described in (Verdult and Verhaegen, 2004).

5. NUMERICAL EXAMPLES

Let us consider the case of an LFPV system composed of 5 single input-single output local LTI systems ($m = 5$). Each local LTI system has two conjugate complex poles with modulus randomly drown in the interval $[0.8, 0.9]$, and a real zero randomly drawn in $[-0.8, 0.8]$. The models of the local LTI system are converted to the state-space form, each with an arbitrary state basis, before being linked together to form an LFPV system model, which is then used to data simulation.

The *estimation data set* is generated with a piecewise-constant scheduling variable sequence $p(t) \in P = \{1, 2, 3, 4, 5\}$ as shown in Figure 1, and an independent random input sequence uniformly distributed in $[0, 1]$. The output is then simulated with the randomly generated LFPV model as described above. No state noise is added during the simulation, but a white Gaussian noise with standard deviation 0.01 is added to the simulated output.

From one randomly generated estimation data set, 5 independent local LTI models of second order ($n_x = 2$) are first estimated with the PEM (Ljung, 1999), which are then transformed into a coherent set of LTI models with the method presented in this paper. The LFPV model composed of the 5 coherent local LTI models is then tested on an evaluation data set, which is generated with an *independent* random scheduling variable sequence $p(t)$ equally distributed within $P = \{1, 2, 3, 4, 5\}$, which is *radically different* from the one used in the estimation data generation, as shown in Figure 2. The output simulated with the estimated LFPV model is then compared with the true output in Figure 2, where the two curves (blue and green) are hardly distinguishable. The model fit (percentage of the output variance explained by the estimated model) in this example is 0.9537.

The above result is only based on one randomly generated estimation data set and one validation set. The same simulation is then randomly repeated 1000 times, with different random realizations of the LFPV system composed of local

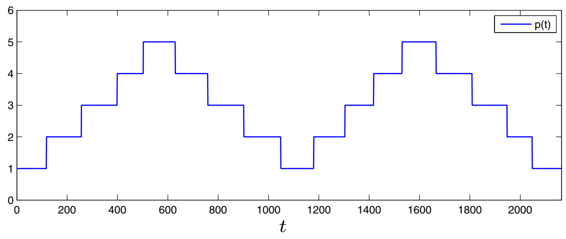

Fig. 1. Scheduling sequence $p(t)$ of the estimation data set.

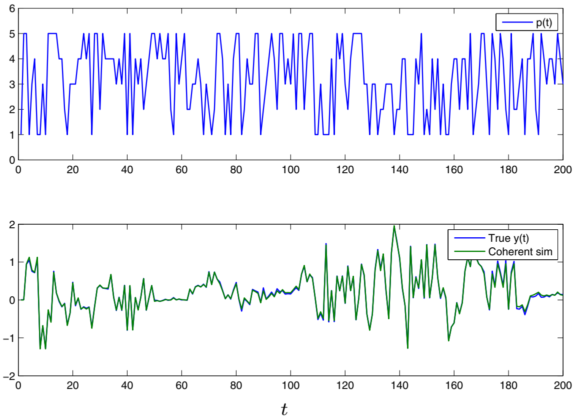

Fig. 2. Top: scheduling sequence $p(t)$ of the validation data set. Bottom: comparison between the true output (blue) and the output simulated with *coherent* local LTI models (green).

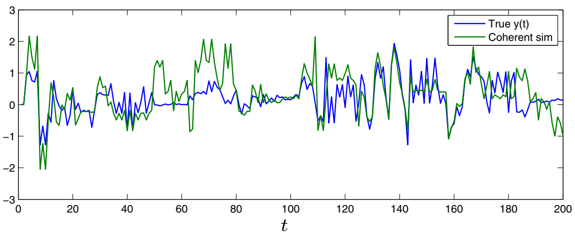

Fig. 3. Comparison between the true output (blue) and the output simulated with *non coherent* local LTI models (green).

LTI models and different random noise realizations. Based on these results, the empirical mean of the model fit is 0.9639, and standard deviation 0.0306.

In order to illustrate the importance of making estimated local LTI models coherent, for the same random realization as shown in Figure 2, the output simulated with the estimated local LTI models before their "coherentization" is compared to the true output in Figure 3. The benefit of coherent models is then clear.

6. CONCLUSION

For the purpose of LPV system identification, a data-based method has been proposed in this paper to make independently estimated local LTI models coherent, without making any global structural assumption about the LPV system, based on an algorithm initially introduced in (Verdult and Verhaegen, 2004).

As a final note, because the local model interpolation problem, as studied in (De Caigny et al., 2011, 2014), is not considered in the present paper, it does not matter if the elements p_i of the finite set P are scalar real values or any other mathematical objects. Throughout this paper, the values p_i could have been replaced by their indexes i, defined in any arbitrary order.

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